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The periodic Toda chain and a matrix generalization of the Bessel function recursion relations

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Abstract. We obtain the quantization conditions of the periodic Toda lattice in the Baxter form:

$$\Lambda(u)Q(u) = \mathrm{i}^{N}Q(u+\mathrm{i}\hbar) + \mathrm{i}^{-N}Q(u-\mathrm{i}\hbar)$$

Λ is the 'transfer matrix' containing the information about the spectrum and Q is an integral operator commuting with Λ. The logarithms of the matrix elements of Q are the generating functions of the canonical Bäcklund transformation. The requirement that Q is analytic and vanishes when u goes to infinity completely determines the spectrum of Λ.

The Toda lattice [1–4] is a one-dimensional chain of equal masses with exponential interactions between nearest neighbours. When the chain is finite, either the first and last masses are decoupled (the open chain) or they are coupled together (the periodic chain). Both systems are completely integrable in the sense that one can construct as many constants of the motion as they possess degrees of freedom. The two mechanical systems however, behave quite differently. The open chain has a continuous spectrum while the periodic chain has quantum states and a discrete spectrum.

In this paper, we are concerned with the determination of the spectrum of the periodic chain. This problem was considered by Gutzwiller [5] who separated the variables and derived recursion relations of the type:

$$-Q_{v-1} + Q_{v+1} = \Lambda(v)\ Q_v \tag{1}$$

where $\Lambda(v)$ is a polynomial whose coefficients are the unknown constants of the motion. Sklyanin [6, 7] greatly simplified the derivation of (1) using the R matrix formalism. Moreover, he suggested to interpret it as a Bethe ansatz equation defining an analytical function $Q(v)$.

In this paper, we derive (1) using the methods of statistical mechanics [9]. $\Lambda(v)$ is the 'transfer matrix' and Q is an integral operator commuting with Λ. The matrix elements of Q turn out to be the exponential of the generating function of the canonical Bäcklund transformation [1, 4]. Diagonalizing simultaneously Λ and Q, we recover (1) as an equation for their eigenvalues. The requirement that Q is entire

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and goes to zero when v goes to infinity in the imaginary direction determines both Q and the polynomial Λ. It generalizes to higher degrees the case of degree one $\Lambda = v$ where it is known that the unique solution of (1) vanishing in the imaginary direction is the Bessel function K_v† [8] considered as a function of the index.

The equations of motion of the periodic Toda lattice derive from the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N} \mathrm{e}^{(q_{i+1}-q_i)} \tag{2}$$

where the index i is defined modulo N. They take the form

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} = p_i \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} = \mathrm{e}^{q_{i-1}-q_i} - \mathrm{e}^{q_i - q_{i+1}}. \end{aligned} \tag{3}$$

Their integrability results from the following Lax pair representation [4, 6]: Define matrices L_i and M_i by

$$\begin{aligned} L_i &= \begin{pmatrix} u - p_i & \mathrm{e}^{q_i} \\ -\mathrm{e}^{-q_i} & 0 \end{pmatrix} \\ M_i &= \begin{pmatrix} u & \mathrm{e}^{q_i} \\ -\mathrm{e}^{-q_{i+1}} & 0 \end{pmatrix}. \end{aligned} \tag{4}$$

The system of equations (3) is equivalent to the auxiliary problem:

$$\dot{L}_i = M_{i-1} L_i - L_i M_i. \tag{5}$$

Consider the monodromy matrix

$$T(u) = L_1(u) L_2(u) \cdots L_N(u). \tag{6}$$

From (2)

$$\dot{T}(u) = [M_N, T(u)]. \tag{7}$$

Therefore, the trace of $T(u)$, $\Lambda(u)$ is independent of the time. It is a polynomial of degree N in u whose coefficients are the constants of motion

$$\Lambda(u) = u^N - P\, u^{N-1} + \left(\frac{P^2}{2} - H\right) u^{N-2} + \cdots. \tag{8}$$

P is the momentum, H the Hamiltonian.

In quantum mechanics, the p_i are replaced by the operators $\frac{\hbar}{i}\frac{\partial}{\partial q_i}$ so that the matrix elements of L_i do not commute. Their commutation relation can be expressed

† Our convention is $K_v(z) = (1/\sin \pi v)\,(I_v(z) - I_{-v}(z))$.

as follows [4, 6]. Define the matrix $\overset{1}{L}$ and $\overset{2}{L}$ to be respectively $L \otimes 1$ and $1 \otimes L$ and the 4×4 R matrix

$$R(u) = \begin{pmatrix} u - \mathrm{i}\hbar & \cdot & \cdot & \cdot \\ \cdot & u & -\mathrm{i}\hbar & \cdot \\ \cdot & -\mathrm{i}\hbar & u & \cdot \\ \cdot & \cdot & \cdot & u - \mathrm{i}\hbar \end{pmatrix} \tag{9}$$

Then

$$R(u_1 - u_2)\, \overset{1}{L}_i(u_1)\, \overset{2}{L}_i(u_2) = \overset{2}{L}_i(u_2)\, \overset{1}{L}_i(u_1)\, R(u_1 - u_2)\,. \tag{10}$$

L_i commutes with L_j if $i \neq j$. The commutation relations of the matrix elements of $T(u)$ follow in a straightforward way from (10) and are given by the same expression

$$R(u_1 - u_2)\, \overset{1}{T}(u_1)\, \overset{2}{T}(u_2) = \overset{2}{T}(u_2)\, \overset{1}{T}(u_1)\, R(u_1 - u_2)\,. \tag{11}$$

It follows that the $N-1$ coefficients of the trace of T, $\Lambda(u)$, are conserved quantities in involution.

To diagonalize $\Lambda(u)$, we shall adapt the method used by Baxter in the eight-vertex model case [9]. We construct a family of integral operators $Q(u)$ which satisfy the matrix relation:

$$\Lambda(u)Q(u) = \mathrm{i}^N Q(u + \mathrm{i}\hbar) + \mathrm{i}^{-N} Q(u - \mathrm{i}\hbar) \tag{12}$$

and such that $Q(u)$, $Q(v)$, $\Lambda(v)$ commute for all values of u and v.

In (12), Q is a matrix with rows and columns indexed by the (continuous) variables $(q_1, \dots, q_N)$, $(q'_1, \dots, q'_N)$. As a first step to finding the solution, we consider the equation for the columns of Q, $y_u(q_1, \dots, q_N)$. We take y in the form of a direct product

$$y(q_1 \cdots q_N) = \prod_{i=1}^{N} \varphi_i(q_i) \tag{13}$$

so that the product Λy takes the form:

$$\Lambda(u) y = \operatorname{tr}\,(L_1 \varphi_1) \dots (L_N \varphi_N)\,. \tag{14}$$

The product Λy decomposes into two terms $y' + y''$ if each of the matrices $L_j \varphi_j$ is lower triangular. Due to the cyclicity of the trace, Λ is not modifed if we substitute $\tilde{L}_j = N_j L_j N_{j+1}^{-1}$ to L_j in (6). We take N_j of the form

$$N_j = \begin{pmatrix} 1 & \mathrm{i}\,\mathrm{e}^{q'_j} \\ 0 & 1 \end{pmatrix} \tag{15}$$

and equate to zero the upper right coefficient of $\tilde{L}_j \varphi_j$; this gives

$$\left(p_j + \frac{1}{\mathrm{i}} \mathrm{e}^{q_j - q'_{j+1}} + \mathrm{i}\mathrm{e}^{q'_j - q_j} - u \right) \varphi_j = 0 \tag{16}$$

which is solved by

$$\varphi_j(u) = \exp\left(\frac{1}{\hbar}\left(\mathrm{i}u\left(q_j - q'_{j+1}\right) - \mathrm{e}^{q_j - q'_{j+1}} - \mathrm{e}^{q'_j - q_j}\right)\right) \tag{17}$$

and

$$\tilde{L}_j\varphi_j = \begin{pmatrix} -\mathrm{i}\,\varphi_j(u-\mathrm{i}\hbar) & 0 \\ * & \mathrm{i}\,\varphi_j(u+\mathrm{i}\hbar) \end{pmatrix}. \tag{18}$$

It follows directly from (18) that (12) is satisfied with y substituted for Q. Let us define the kernel:

$$\begin{aligned} Q_u(q \mid q') &= \exp\frac{1}{\hbar}\left(\mathrm{i}u\left(\sum_{j=1}^{N} q_j - \sum_{j=1}^{N} q'_j\right) - \sum_{j=1}^{N}\left(\mathrm{e}^{q'_j - q_j} + \mathrm{e}^{q_j - q'_{j+1}}\right)\right) \\ &= \prod_{j=1}^{N} W_u\left(q'_j - q_j\right)\bar{W}_u\left(q_j - q'_{j+1}\right) \end{aligned} \tag{19}$$

with

$$\begin{aligned} W_u(q) &= \exp\frac{1}{\hbar}\left(-\frac{\mathrm{i}u}{2}q - \mathrm{e}^q\right) \\ \bar{W}_u(q) &= \exp\frac{1}{\hbar}\left(\frac{\mathrm{i}u}{2}q - \mathrm{e}^q\right). \end{aligned} \tag{20}$$

By construction, Q satisfies equation (12) and it follows from a similar analysis that it also satisfies

$$Q(u)\Lambda(u) = \mathrm{i}^N Q(u+\mathrm{i}\hbar) + \mathrm{i}^{-N} Q(u-\mathrm{i}\hbar). \tag{21}$$

Note that the logarithms of the matrix elements of Q are the generating functions of the canonical transformation [1, 4]. It may be useful to visualize Q as shown in figure 1.

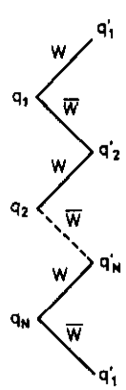

Figure 1. Visualization of matrix Q.

We can repeat the argument of Baxter [9] to show that operators $Q(u)$ with different spectral parameters u commute. Let us introduce the permutation operator C:

$$(Cf)(q_1, q_2, \ldots, q_N) = f(q_2, q_3, \ldots, q_1) \tag{22}$$

and the kernel $\hat{Q}$

$$\begin{aligned} Q_u(q \mid q') &= (Q_u C)(q \mid q') = (C Q_u)(q \mid q') \\ &= \prod_{j=1}^{N} W_u(q'_j - q_{j+1}) \bar{W}_u(q_j - q'_j) . \end{aligned} \tag{23}$$

The equality

$$\hat{Q}(u)Q(v) = \hat{Q}(v)Q(u) \tag{24}$$

is realized if there exists functions $A_u(q)$ which satisfy the identity

$$\begin{aligned} A_{u-v}(q_1 - r_1) &\int_{-\infty}^{+\infty} \mathrm{d}q \; \bar{W}_u(q_1 - q) W_u(q - q_2) W_v(r_1 - q) \bar{W}_v(q - r_2) \\ &= A_{u-v}(q_2 - r_2) \\ &\quad \times \int_{-\infty}^{+\infty} \mathrm{d}q \; \bar{W}_v(q_1 - q) W_v(q - q_2) W_u(r_1 - q) \bar{W}_u(q - r_2) . \end{aligned} \tag{25}$$

for all values of u, v, q_1, q_2, r_1, r_2 (as shown diagrammatically in figure 2).

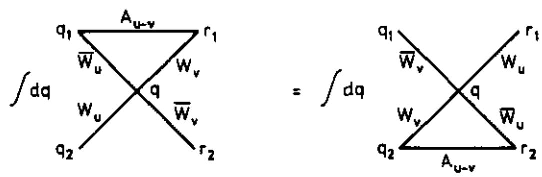

Figure 2. Diagrammatical representation of $A_u(q)$.

A simple calculation gives

$$A_u(q) = (\cosh q/2)^{\frac{iu}{\hbar}} . \tag{26}$$

Now, for u real, the operators $Q(u)$ and $\hat{Q}(u)$ are Hermitian conjugates and commute with C. Therefore, there exists a unitary operator D *independent of* u which diagonalizes $Q(u)$ simultaneously for all values of u. Moreover, in the basis of momentum eigenstates, the matrix elements of Q vanish like $\exp(-\pi N|u|/2)$ when u goes to infinity on the real line. Multiplying (12) by D to the right and D^{-1} to the left, we obtain an equation for the eigenvalue matrices Q_d and Λ_d. The eigenvalue matrix $Q_d(u)$ is entire and vanishes when u tends to infinity in the real direction.

We now consider (12) as a scalar equation and argue that $\Lambda(u)$ is completely determined by the requirement that Q is an entire function going to zero sufficiently

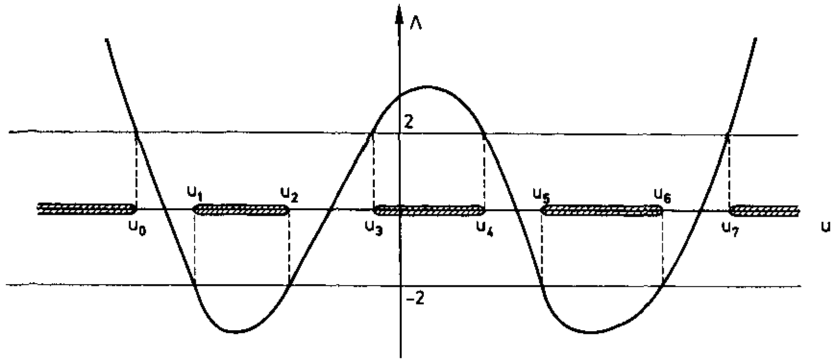

Figure 3. The polynomial $\Lambda(u)$. Shaded, the intervals where $|\Lambda| \geqslant 2$.

fast when $\pm u$ goes to infinity. We first obtain this result in a WKB approximation, then we show how the quantization conditions obtained by Gutzwiller result from these requirements.

Let us look for a solution of (12) in the form

$$Q(u) = \exp\left(-\frac{1}{\hbar}\left(\mathrm{i}\ S(u) + \frac{N\pi}{2}u\right)\right). \tag{27}$$

This gives

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}S(u+\mathrm{i}\hbar)} + \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}S(u-\mathrm{i}\hbar)} = \Lambda(u)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}S(u)}. \tag{28}$$

We expand S in powers of $\hbar$

$$S(u) = S_0(u) + \frac{\hbar}{\mathrm{i}}S_1(u) + \cdots \tag{29}$$

and develop (28) to order 1 to obtain

$$2\cosh S_0' = \Lambda(u) \tag{30}$$

$$S_1' = \tfrac{1}{2}S_0''\tanh S_0' \tag{31}$$

which yields formally

$$Q(u) = \frac{1}{\sqrt{\sinh S_0'(u)}}\exp\left(-\frac{\mathrm{i}}{\hbar}\int^u S_0'(p)\mathrm{d}p\right). \tag{32}$$

At this point, we must determine the branches of the phase $S_0(u)$. For this, we make the assumption that the zeros of $Q(u)$ accumulate on the intervals $|\Lambda(u)| \geqslant 2$ on the real axis. The $N-1$ intervals not containing $\pm\infty$ are called intervals of instability and coincide with the regions where the 'classical motion' of u is confined [2, 3] (figure 3). We therefore take the system of cuts defined by $|\Lambda| \geqslant 2$, that is it to say $[-\infty, u_0]$, $[u_1, u_2]$, $[u_3, u_4]$, $[u_5, u_6]$, $[u_7, +\infty]$, on figure 3. We choose the determination of S_0' so that Q is exponentially decreasing when u goes to plus or minus infinity. The resulting conformal mapping $S_0'(u)$ is represented in figure 4. S_0' is a continuous function of u except across the cuts. For Q to define a uniform function of u, $S(u)$ must be defined modulo $2\pi\hbar$ in the complex plane minus the cuts. This gives the conditions:

$$\int_{C_k} S'(u)\mathrm{d}u = 2\pi\hbar n_k \qquad 1 \leqslant k \leqslant N-1 \tag{33}$$

where the C_k are contours of integration encircling the intervals of instability $|\Lambda| \geqslant 2$. On these intervals, Q is approximated by $Q^{WKB}(u+\mathrm{i}o) + Q^{WKB}(u-\mathrm{i}o)$. The $n-1$ integers n_k count the number of zeros of Q on the kth interval of instability. To first order in $\hbar$, (33) gives the quantization conditions:

$$\int_{u_{2k-1}}^{u_{2k}} \cosh^{-1}\left|\frac{\Lambda}{2}(p)\right| \mathrm{d}p = \pi\hbar\left(n_k + \frac{1}{2}\right). \tag{34}$$

Such conditions are precisely what one would expect from the correspondence principle applied to the solution of the Hamilton–Jacobi equation [3]:

$$S(u_1, \ldots, u_{N-1}, q_N, t) = \sum_{k=1}^{N-1} \int^{u_k} \cosh^{-1}\left(\frac{|\Lambda(p)|}{2}\right) \mathrm{d}p + Pq_N - Et \tag{35}$$

where the real variables u_k are constrained to move on the $N-1$ intervals of instability. The solution so constructed vanishes as $\exp((-N\pi/2\hbar)|u|)\sin((Nu/\hbar)\log Nu)$ when u goes to $\pm\infty$.

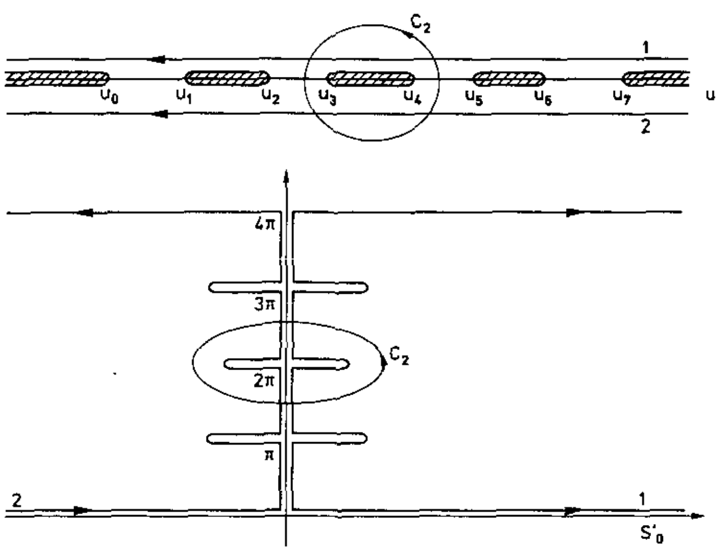

Figure 4. The conformal mapping $S_0'(u)$.

We now look for a solution of (12) which has the same assymptotics as the semiclassical approximation and obtain the quantization conditions in terms of a Hill determinant [5, 8]. We consider the recursion relation satisfied by the auxiliary function

$$\varphi(u) = \prod_{k=1}^{N} \sinh\, \pi\left(\frac{u}{\hbar} + \delta_k\right)\; Q(u) \tag{36}$$

where δ_k are unknown real numbers. It admits two independent entire solutions φ_+ and φ_- which tend to zero when u goes to plus or minus $\mathrm{i}\infty$ respectively, and which increase exponentially like $\mathrm{e}^{(N\pi/2\hbar)|u|}$ in the real direction. We then ask for the function:

$$Q(u) = \frac{\varphi_+(u) - \lambda\varphi_-(u)}{\prod_{k=1}^{N} \sinh\pi((u/\hbar) + \delta_k)} \tag{37}$$

to be regular. This determines the coefficients δ_k and produces the quantization conditions. Due to the denominator, (37) has the correct behaviour $\exp -(N\pi/2\hbar)|u|$ at infinity.

Let us substitute (36) in (12) and set

$$\begin{aligned} u &= \mathrm{i}\hbar v \\ P(v) &= \left(-\frac{\mathrm{i}}{\hbar}\right)^N \Lambda(\mathrm{i}\hbar v). \end{aligned} \tag{38}$$

We obtain the following recursion relations for φ:

$$\varphi(v-1) + (-)^N \varphi(v+1) = \hbar^N P(v)\varphi(v). \tag{39}$$

Equation (37) has two independent solutions φ_+, φ_- distinguished by their asymptotic behaviour. We set

$$\varphi_\pm(v) = \theta_\pm(v)/\pi_\pm(v) \tag{40}$$

with

$$\begin{aligned} \pi_+(v) &= \prod_{k=1}^{N} \hbar^v \Gamma(v + 1 - \mathrm{i}u_k) \\ \pi_-(v) &= \prod_{k=1}^{N} \hbar^{-v} \Gamma(1 - v + \mathrm{i}u_k) \end{aligned} \tag{41}$$

where $\mathrm{i}u_k$ are the roots of the polynomial $P(v) = \prod_1^N (v - \mathrm{i}u_k)$. $\theta_\pm$ are the solutions of the following recursion relations:

$$\begin{aligned} \theta_+(v-1) &= \theta_+(v) + \frac{\theta_+(v+1)}{P(v)P(v+1)} \\ \theta_-(v+1) &= \theta_-(v) + \frac{\theta_-(v-1)}{P(v)P(v-1)} \end{aligned} \tag{42}$$

determined so that $\theta_+(+\infty) = \theta_-(-\infty) = 1$. So, φ_+ defines an entire function which vanishes when v goes to $+\infty$ and φ_- an entire function which vanishes when v goes to $-\infty$. They both increase as $\exp(\frac{1}{2}\pi N|v|)$ for v very large in the imaginary direction. To obtain Q with the correct asymptotic behaviour, we look for a linear combination of φ_+ and φ_- divisible by $\prod_{k=1}^{N} \sin\pi(v - \mathrm{i}\delta_k)$. This can be achieved if two conditions are satisfied. First, there must exist N real numbers δ_k such that for $v = \mathrm{i}\delta_k$ modulo an integer, the two solutions φ_+ and φ_- are proportional, and second, the proportionality coefficient $\lambda = \varphi_+(\mathrm{i}\delta_k)/\varphi_-(\mathrm{i}\delta_k)$ must be independent of k.

The first condition is realized if the Wronskian

$$W(\varphi_+, \varphi_-) = \varphi_+(v+1)\varphi_-(v) - \varphi_+(v)\varphi_-(v+1) \tag{43}$$

vanishes for v equal to $\mathrm{i}\delta_k$ modulo an integer.

A direct calculation gives

$$W(\varphi_+, \varphi_-) = \theta(v) \prod_{k=1}^{N} \pi^{-1} \sin \pi(v - \mathrm{i}u_k) \tag{44}$$

where $\theta(v)$ is the infinite tridiagonal determinant of Hill's [8]:

$$\theta(v) = \begin{vmatrix} \ddots & \ddots & \ddots & 0 \\ -\frac{1}{P(v)} & 1 & \frac{1}{P(v)} & \\ & -\frac{1}{P(v+1)} & 1 & \frac{1}{P(v+1)} \\ 0 & \ddots & \ddots & \ddots \end{vmatrix}. \tag{45}$$

To evaluate the v dependence of θ, one observes that θ is analytic except at the roots of P, $\mathrm{i}u_k$, where it has simple poles. It is periodic with period 1 and tends to 1 when v tends to infinity in the imaginary direction. It follows that

$$\theta(v) = 1 + \sum_{k=1}^{N} \varepsilon_k \cot \pi(v - \mathrm{i}u_k) = \prod_{k=1}^{N} \frac{\sin \pi(v - \mathrm{i}\delta_k)}{\sin \pi(v - \mathrm{i}u_k)} \tag{46}$$

with $\sum_k u_k = \sum_k \delta_k$ and $\sum_k \varepsilon_k = 0$. Setting θ to zero determines the δ_k in terms of the N residues ε_k of $\theta(v)$ at $v = \mathrm{i}u_k$.

The second condition yields the quantization conditions:

$$\frac{\varphi_+(\mathrm{i}\delta_1)}{\varphi_-(\mathrm{i}\delta_1)} = \frac{\varphi_+(\mathrm{i}\delta_2)}{\varphi_-(\mathrm{i}\delta_2)} = \cdots = \frac{\varphi_+(\mathrm{i}\delta_N)}{\varphi_-(\mathrm{i}\delta_N)}. \tag{47}$$

Assuming that δ_k, u_k are real, $\varphi_+(\mathrm{i}\delta_k)$ and $\varphi_-(\mathrm{i}\delta_k)$ are complex conjugates. Therefore (47) defines N phases that must be equal. These are precisely the quantization conditions obtained by Gutzwiller from a different point of view.

To conclude, the eigenfunctions of the commuting set of operators $Q(u)$ are generalizations of the modified Bessel function $K_{\mathrm{i}u}$ which occur in the lowest degree case, $\Lambda(u) = \mathrm{i}u$. In that case, there is only one value of δ equal to zero and no quantization condition (47).

Acknowledgments

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